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Selective withdrawal from a stably stratified fluid

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In this paper a reservoir connected through a horizontal contraction to a channel is considered. Both the reservoir and the channel are considered to contain a stable multi-layered system of fluids. The conditions under which there is a flow in only one layer, and the depth in this flowing layer decreases continuously from its depth in the reservoir to its depth in the channel, give the maximum discharge that can be obtained with a flow only from this single layer. For this case the volume discharge calculations are carried out at a single section (the section of minimum width). Where there are velocities in only two layers and the depth in each of these layers decreases continuously from their depths in the reservoir to their depths in the channel, the theory involves computations at two sections in the flow. These are the section of minimum width and a section upstream of the position of minimum width (the virtual point of control). For this flow it is shown that the solution is the one in which the velocity and density distributions are self similar and that the depths of the layers at the point of maximum contraction are two-thirds of those far upstream. It is then shown that for any stable continuous or discrete density stratification in the reservoir a self similar solution will satisfy the conditions for the depths of the flowing layers to decrease smoothly from the reservoir to downstream of the contraction. Again the ratio of the depth at the contraction to that far upstream is two-thirds.

When there is a very large density difference between the fluid in the lower dead water and that in the lowest flowing streamline then this streamline becomes horizontal and may be considered as a frictionless bed. The flow when the bed is not horizontal but where there is a small rise in the channel at the position of maximum contraction is considered for the case where two discrete layers flow under a volume of dead water. In this case the velocity and density profiles are not self similar.

Experiments have been carried out with a contraction in a flume for the withdrawal of two discrete layers from a three layer system and the withdrawal from a fluid with a linear density gradient. In both cases the reservoir and channel bed and hence the lowest streamline was effectively horizontal. These experiments confirmed the theoretical predictions.

1. Introduction

The problem of the selective withdrawal of a fluid of known density from a stably stratified fluid in a reservoir is one of obvious practical importance and this paper makes a contribution towards the understanding of this problem for the case where the fluid is continuously stratified or consists of more than two discrete layers. In a natural reservoir vertical density gradients may arise from variations in temperature, dissolved salt content, and suspended sediment loads. In a number of these situations it is important to withdraw fluid of known properties and the relationship between the properties of the discharged fluid and the rate of discharge is required.

In power station cooling ponds, control structures are designed so that only the coolest water is used. In this case the pond most frequently consists of two distinct layers and it is therefore not surprising that the first investigations were for the withdrawal of one layer from a two layer system. Craya (1949) considered the case of a horizontal line sink located a distance above an interface between two fluids. Then using the method of conformal transformation he was able to obtain the discharge below which the fluid in the lower layer remained stagnant in terms of a critical Froude number. Craya's calculation agrees with the experimental results obtained by Gariel (1949).

Harleman, Morgan & Purple (1959) have performed similar experiments for the axisymmetric case with a sink at the bottom of a large tank containing two fluid layers. In a similar fashion the results they obtained for the discharge below which the upper layer remained stagnant can be expressed in terms of a critical Froude number. In both of the above cases the velocity distributions in the flowing layer far upstream from the sink were uniform.

Where the fluid is stratified in more than one layer or is continuously stratified the problem is more difficult. When the reservoir is of limited horizontal extent some of the fluid layers have a limited volume and thus as withdrawal takes place the upstream conditions change and hence the flows must be time dependent. For the case of a reservoir of effectively infinite horizontal extent and of a given density variation at infinity, then, if the discharge is fixed it is not known whether the withdrawal is from a relatively deep layer with a small velocity, or from a shallow layer with a higher velocity. Further, the velocity distribution at infinity is not known.

For the case where viscous effects limit the flow and where the fluid is weakly stratified with a linear density gradient Koh (1966) obtained a solution in which the velocity and density profiles were self similar both for the axisymmetric and the two-dimensional cases. The two-dimensional case was treated as a quasi-steady flow and for this case experiments confirmed the theoretical predictions.

Yih (1965, 1958) examined the two-dimensional flow of a stratified fluid into a line sink which is between two horizontal planes. He assumed that the density gradient far upstream was linear and that the velocity distribution far upstream was given by $[\rho]^{\frac{1}{2}}u$ equals a constant (where ρ is the density of the fluid and u is its horizontal velocity). He points out that 'if the flow issues horizontally from a large reservoir $[\rho]^{\frac{1}{2}}u$ is indeed a constant'. With this boundary condition and assuming an inviscid fluid a complete solution was obtained in terms of an upstream Froude number for Froude numbers of greater than $1/\pi$. Kao (1965) has recently extended this approach to Froude numbers of less than $1/\pi$.

When a reservoir containing a stably stratified fluid is connected to a channel and a band of this stratified fluid is withdrawn (the remaining portions of the fluid being stationary) then it will be shown that gravitational effects are important in the region between the reservoir and the channel. These effects cause a velocity distribution in the channel which is significantly different from that described above.

Ideally, we would like to be able to solve the problem of the withdrawal from a point sink situated in an infinite reservoir containing fluid with an arbitrary stable stratification. That is, given a withdrawal rate we need to know the velocity distribution at a distance from the sink in order to estimate the density of the fluid that is being withdrawn. This general problem has to date proved intractable. However, a solution can be obtained when the fluid is withdrawn through a smoothly contracting channel in which there is a definite minimum width (figure 1). For this case it is reasonable to make the hydrostatic approximation and to use arguments that are extensions of the one-dimensional ones used in the one layer systems of open channels. The withdrawal through the contraction of fluid from two adjacent layers of a multilayer system is the simplest case and will be considered first.

It is important to note that the theory considers steady flows only and these would be difficult to obtain in a practical situation. However, the theory should be satisfactory provided that the reservoir is sufficiently large so that the time for a particle to travel through the contraction to the outlet is short compared to the time for the streamline patterns to change due to the withdrawal of fluid from the reservoir.

2. The case of two flowing layers

Consider the system illustrated in figure 1. It is required to find the conditions under which there is a flow only in layers 1 and 2 and the depths in these layers decrease smoothly from their depths in the reservoir to their depths in the channel. Below and above these layers, the fluid is stationary and may consist either of fluid of a single density or a multilayer system of fluids. Consider the fluid above and below the flowing layers to be of a constant density.

Let the density of layers $0, 1, 2, 3$ be $\rho_0, \rho_1 = \rho_0 + \Delta\rho_1$, $\rho_2 = \rho_0 + \Delta\rho_1 + \Delta\rho_2$ and $\rho_3 = \rho_0 + \Delta\rho_1 + \Delta\rho_2 + \Delta\rho_3$, respectively and the depth in the reservoir of the layers $1, 2, 3$ be Y_1, Y_2 and Y_3. Further let the depths of the layers at any point in the contraction be y_1, y_2 and y_3 and velocities and discharges in layers 1 and 2 be v_1, v_2 and Q_1, Q_2 respectively. Then the respective Bernoulli equations for the flowing layers may be written as

$$\frac{1}{2}\frac{\rho_1}{\Delta\rho_1}\frac{v_1^2}{g} + y_1 + y_2 + y_3 = Y_1 + Y_2 + Y_3, \tag{1}$$

$$\frac{1}{2}\frac{\rho_2}{\Delta\rho_2}\frac{v_2^2}{g} + \frac{\Delta\rho_1}{\Delta\rho_2}y_1 + \left(1 + \frac{\Delta\rho_1}{\Delta\rho_2}\right)(y_2 + y_3) = \frac{\Delta\rho_1}{\Delta\rho_2}Y_1 + \left(1 + \frac{\Delta\rho_1}{\Delta\rho_2}\right)(Y_2 + Y_3), \tag{2}$$

and the condition that there is no flow in the lowermost layer is

$$\frac{\Delta\rho_1}{\Delta\rho_3}y_1 + \left(\frac{\Delta\rho_1 + \Delta\rho_2}{\Delta\rho_3}\right)y_2 + \left(1 + \frac{\Delta\rho_1 + \Delta\rho_2}{\Delta\rho_3}\right)y_3$$
$$= \frac{\Delta\rho_1}{\Delta\rho_3}Y_1 + \left(\frac{\Delta\rho_1 + \Delta\rho_2}{\Delta\rho_3}\right)Y_2 + \left(1 + \frac{\Delta\rho_1 + \Delta\rho_2}{\Delta\rho_3}\right)Y_3. \tag{3}$$

If we define $\alpha_{nm} = \Delta\rho_n/\Delta\rho_m, Y'_n = Y_n/Y_1, y'_n = y_n/Y_1$ and use equation (3) to eliminate y_3 then (1) and (2) become

$$\frac{1}{2}\frac{\rho_1}{\Delta\rho_1 g}\left(\frac{Q_1}{by_1}\right)^2\frac{1}{Y_1} + Ay'_1 + By'_2 = A + BY'_2, \tag{4}$$

$$\frac{1}{2}\frac{\rho_2}{\Delta\rho_2 g}\left(\frac{Q_2}{by_2}\right)^2\frac{1}{Y_1} + Cy'_1 + Dy'_2 = C + DY'_2, \tag{5}$$

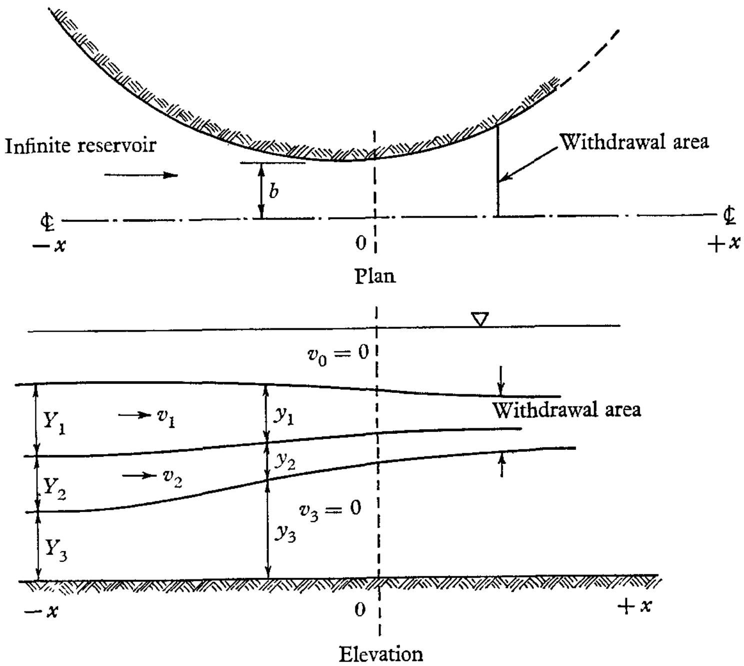

FIGURE 1. The two layer system.

where b is the channel width and

$$A = (1+\alpha_{23})/(1+\alpha_{13}+\alpha_{23}),$$
$$B = 1/(1+\alpha_{13}+\alpha_{23}),$$
$$C = \alpha_{12}/(1+\alpha_{13}+\alpha_{23}),$$
$$D = (1+\alpha_{12})/(1+\alpha_{13}+\alpha_{23}).$$

If Q_1, Q_2 and the conditions at infinity (Y'_2, A, B, C and D) are known, then (4) and (5) determine y'_1 and y'_2 in terms of b at every x. However, as b decreases smoothly from its large value at minus infinity to its minimum and then increases again it is required that the depth of each layer continuously decrease at a finite rate. This determines the possible flow. It is therefore appropriate to examine the conditions for which dy'_1/dx and dy'_2/dx are always finite. Differentiating (4) and (5), defining

$$F_1^2 = \frac{\rho_1}{g\Delta\rho_1}\frac{Q_1^2}{b^2y_1^3} \quad \text{and} \quad F_2^2 = \frac{\rho_2}{g\Delta\rho_2}\frac{Q_2^2}{b^2y_2^3}$$

and solving for dy_1'/dx and dy_2'/dx, we obtain

$$\frac{dy_1'}{dx} = \frac{1}{b}\frac{db}{dx}\frac{D_2}{D_1}, \tag{6}$$

$$\frac{dy_2'}{dx} = \frac{1}{b}\frac{db}{dx}\frac{D_3}{D_1}, \tag{7}$$

where

$$D_1 = (A-F_1^2)(D-F_2^2)-BC,$$

$$D_2 = \begin{vmatrix} F_1^2 y_1' & B \\ F_2^2 y_2' & D-F_2^2 \end{vmatrix},$$

$$D_3 = \begin{vmatrix} A-F_1^2 & F_1^2 y_1' \\ C & F_2^2 y_2' \end{vmatrix}.$$

Now for smooth transition it is required that D_1 does not equal zero unless db/dx equals zero (the point of control) or D_2 and D_3 equal zero. It is worth emphasizing that the condition D_2 and D_3 equal zero together (the point of virtual control) implies that D_1 also equals zero. It is these two conditions which ultimately enable the relationships between Q_1, Q_2 and Y_1, Y_2 to be obtained.

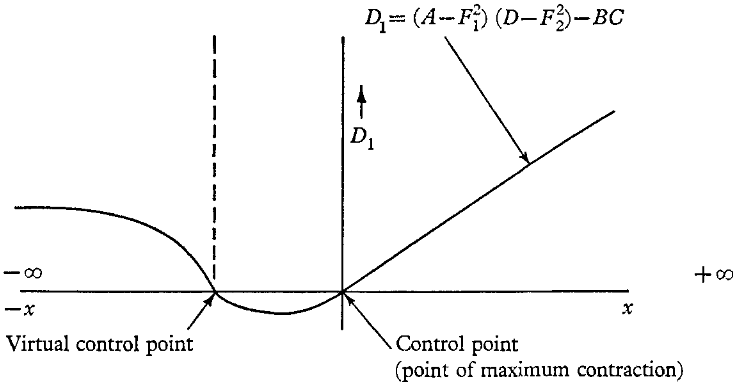

FIGURE 2. The points of control for a two layer system.

Now consider the variation of the determinant D_1. In the reservoir, F_1^2 and F_2^2 will tend to zero and hence $D = AD-BC = 1/(1+\alpha_{13}+\alpha_{23})$ is positive. Far downstream F_1^2 and F_2^2 will tend to large positive values and hence D_1 is again positive. Now it can be shown that when $b(x)$ has only one minimum the graph of D_1 versus x has only one turning point and that for the case where dy_1'/dx and dy_2'/dx are both negative then dD_1/dx at the position of minimum width is positive.

Thus the graph of D_1 versus x is as in figure 2 and the equation

$$D_1 = 0 \tag{8}$$

holds at the section of minimum width and at some section upstream. This latter section will be called the point of virtual control and in order that dy_1'/dx and dy_2'/dx be finite at this section then from (6) and (7)

$$D_2 = 0, \tag{9}$$

$$D_3 = 0. \tag{10}$$

Equations (9), (10), (4) and (5) may be solved for the four unknowns Q_1/by_1, Q_2/by_2, y'_1 and y'_2. These equations are solved using $c^2 = (v_2/v_1)^2$, $\phi = \rho v_1^2/2\Delta\rho_1 g Y_1$, y'_1 and y'_2 as the variables and eliminating y'_1 and y'_2 from equations (4) and (5) and substituting into equations (9) and (10). If ρ_{21} is defined as ρ_2/ρ_1 then the solutions are

$$c^2 = \frac{DY'_2+C}{\rho_{21}\alpha_{12}(A+BY'_2)}, \tag{11}$$

$$y'_2/y'_1 = Y'_2, \tag{12}$$

$$\phi = \frac{1}{2}\frac{\rho_1}{\Delta\rho_1 g}\frac{v_1^2}{Y_1} = \frac{(A+BY'_2)(AD-BC)Y'_2}{(AD-BC)Y'_2+2(C+DY'_2)(A+BY'_2)}, \tag{13}$$

$$y'_1 = \frac{2(C+DY'_2)(A+BY'_2)}{(AD-BC)Y'_2+2(C+DY'_2)(A+BY'_2)} \tag{14}$$

and $$y'_3 = Y'_3 + \frac{\alpha_{13}+(\alpha_{23}+\alpha_{13})Y'_2}{1+\alpha_{13}+\alpha_{23}}\left[\frac{(AD-BC)Y'_2}{(AD-BC)Y'_2+2(C+DY'_2)(A+BY'_2)}\right]. \tag{15}$$

This completes the calculations at the virtual control point and it is worth noting that these equations have determined the velocities and depths, and hence the ratio of the discharge may be calculated from (4), (5), (12) and (13). These yield

$$\left(\frac{Q_2}{Q_1}\right)^2 = \frac{1}{\alpha_{12}\rho_{21}}\left(\frac{y'_2}{y'_1}\right)^2\frac{C(1-y'_1)+D(Y'_2-y'_2)}{A(1-y'_1)+B(Y_2-y'_2)} = \frac{(C+DY'_2)(Y'_2)^2}{\alpha_{12}\rho_{21}(A+BY'_2)} \tag{16}$$

and since the flow is steady this discharge ratio does not change with x. It must be noted, however, that position of the virtual control point is as yet unknown. Further at the section of minimum width db/dx equals zero and hence (8) must be satisfied. Thus using (4) and (5) to eliminate F_1^2 and F_2^2, equation (8) becomes

$$\frac{[A(3y'_1-2)+B(2y'_2-2Y'_2)][C(2y'_1-2)+D(3y'_2-2Y'_2)]}{y'_1 y'_2} = BC \tag{17}$$

and the simultaneous solution of these two equations yields, in addition to the solutions obtained at the virtual control point, $y'_1 = \frac{2}{3}$, $y'_2 = \frac{2}{3}Y'_2$ and

$$y'_3 = Y'_3 + \frac{1}{3}\frac{\alpha_{13}+(\alpha_{23}+\alpha_{13})Y'_2}{1+\alpha_{13}+\alpha_{23}}.$$

These pairs of values of y'_1 and y'_2 then give the two solutions to (8). It now only remains to calculate the ratio of the width at the virtual control point to the width at the point of maximum contraction. This may be done by using the continuity equation for either layer and leads to

$$\frac{b_c}{b_m} = \left(\frac{1}{3}\right)^{\frac{3}{2}}\frac{[Y'_2(AD-BC)+2(C+DY'_2)(A+BY'_2)]^{\frac{3}{2}}(1+\alpha_{23}+Y_2)^{\frac{1}{2}}}{(AD-BC)^{\frac{1}{2}}(A+BY'_2)^{\frac{3}{2}}(C+DY'_2)(Y'_2)^{\frac{1}{2}}(1+\alpha_{13}+\alpha_{23})^{\frac{1}{2}}}, \tag{18}$$

where b_c is the width at point of virtual control and b_m is the minimum width.

Substitution of the solution for the layer depths at the position of the maximum contraction into the Bernoulli equations yields for the discharge in each layer

$$Q_1 = \left[\frac{2}{3}\frac{\Delta\rho_1}{\rho_1} g \left(\frac{1+\alpha_{23}+Y_2'}{1+\alpha_{13}+\alpha_{23}}\right)\right]^{\frac{1}{2}} \tfrac{2}{3} b_c Y_1^{\frac{3}{2}}, \tag{19}$$

$$Q_2 = \left[\frac{2}{3}\frac{\Delta\rho_2}{\rho_2} g \left(\frac{(1+\alpha_{12})Y_2'+\alpha_{12}}{1+\alpha_{13}+\alpha_{23}}\right)\right]^{\frac{1}{2}} \tfrac{2}{3} b_c Y_1^{\frac{1}{2}} Y_2. \tag{20}$$

The most remarkable result that has come from the above calculations is that at both the virtual point of control and at the position of minimum width (the main control) the ratio of the layer depths is the same as that at infinity. Furthermore, the ratio of the velocities in the layers at both points is also constant. This implies that the two layers are behaving as if they were a single layer with a composite density. In view of this it is not surprising that the depths at the position of minimum width should be two thirds of those in the reservoir.

The constancy of velocity and depth ratios implies that the velocity and density profiles when plotted non-dimensionally are similar at all points. This suggests that for any density distribution in the reservoir we might expect the velocity and density profiles in the flowing layer to be self similar. Indeed it is a relatively simple matter to show that for any density stratification in the reservoir a solution with self similar velocity and density profiles satisfies the conditions for a layer of smoothly decreasing depth to flow from the reservoir through the contraction† (§3). It is also worth noting that (i) no Boussinesq type approximation has been used in obtaining these results and hence they should hold for large density differences; (ii) by making the density difference between layers (2) and (3) tend to infinity then the streamline between these layers effectively becomes a frictionless level solid boundary.

Further the virtual point of control can be interpreted in the same manner as is a normal control in open channel flow. At the normal control the velocity of a kinematic or a gravity wave tends to zero. In a two layer system two wave modes are possible. It is a simple matter to show that at the point of control (i.e. the point of minimum width) the velocity of the first wave mode becomes zero and the velocity of the second wave mode is negative; while at the point of virtual control the velocity of the second wave mode is zero and that of the first is positive.

3. The solution for an arbitrary density gradient

Consider a reservoir containing a fluid with a known arbitrary density gradient which may be either a continuous or a discontinuous function of depth. Let the reservoir be connected to a narrow channel as in figure 3. When withdrawal takes place the uppermost and lowermost streamlines will respectively drop down and rise up to the outlet as in figure 3. The wedge shaped regions caused by this change in the constant density lines (the streamlines) will be filled by fluid with a density closest to the density of the outermost streamlines of the

† Benjamin (1967) obtains this similarity solution in a completely different manner.

flowing layers. Now, provided the reservoir area is large and the volume in these wedge-shaped regions is relatively small then the fluid filling each of these regions will come from a relatively small range of depth in the reservoir. Thus, provided the density variation over each small range of depth is not too large, the fluid in both the upper and lower wedge shaped regions can be regarded as of approximately constant density. This implies that the uppermost and lowermost streamlines can be considered as lines of constant pressure.

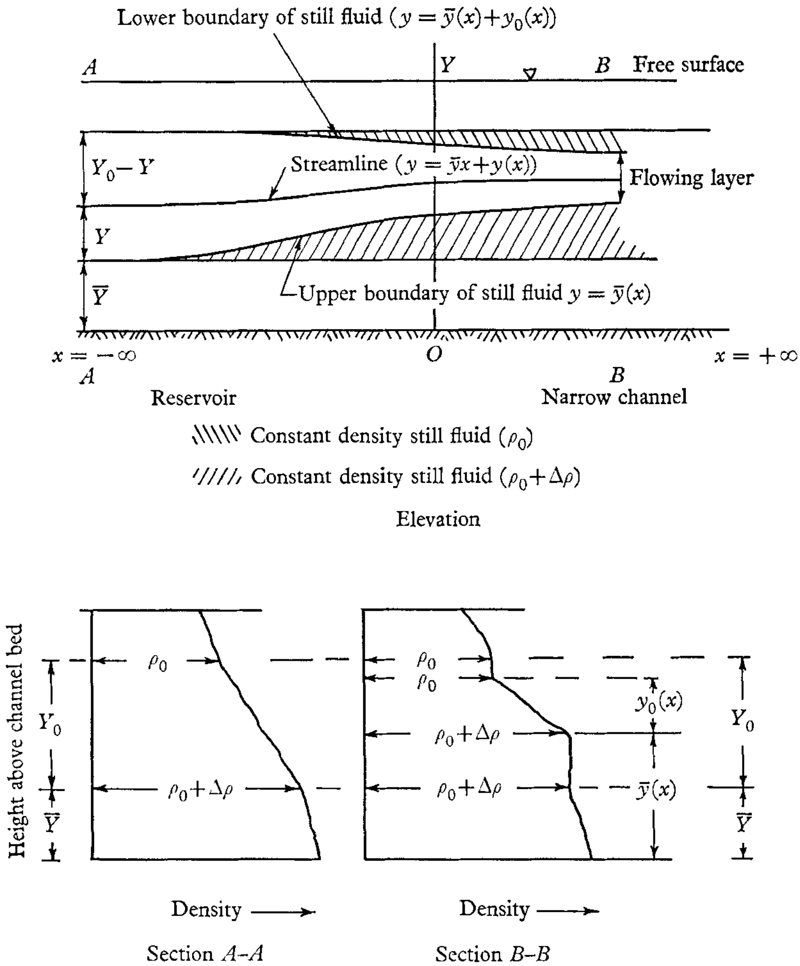

FIGURE 3. The case of an arbitrary density gradient.

Now let the density of the fluid on the streamline which is the upper boundary of the lowermost dead water be $\rho_0 + \Delta\rho$ and let the position of this boundary at $x = -\infty$ and x be given by $\bar{Y}$ and $\bar{y}(x)$ respectively. Then below $y = \bar{Y}$ the density distribution will be unchanged by the flow and as discussed above the density in the lower wedge-shaped region defined by $\bar{Y} \leqslant y \leqslant \bar{y}(x)$ will be $\rho_0 + \Delta\rho$. Similarly, let the density of the fluid on the streamline which is the lowermost boundary of the upper dead water be ρ_0 and let the streamlines position at $x = -\infty$ and at x

be given by $\bar{Y}+Y_0$ and $\bar{y}(x)+y_0(x)$ respectively. Then the density of the fluid in the upper wedge defined by $\bar{y}(x)+y_0(x) \leqslant y \leqslant \bar{Y}+Y_0$ is ρ_0 and above $y = \bar{Y}+Y_0$ the density distribution will be unchanged.

Consider a streamline within the flowing layer. Let its position at $x = -\infty$ and x be respectively $\bar{Y}+Y$ and $\bar{y}(x)+y(x)$ and let the density on this streamline be $\rho_0+\Delta\rho_y$. Let the velocity on this streamline be $v_y(x)$.

It is remarkable that a solution exists in which the distribution of density and velocity are self similar. Let the distribution of density at all sections have the form

$$\frac{\Delta\rho_y}{\Delta\rho} = f\left(\frac{y}{y_0}\right) = f(\eta),$$

where $\eta = y/y_0$ and $f(\eta)$ is determined by the upstream density distribution. Now the flow is steady and the density is constant along a streamline so the constancy of the non-dimensional form of the density distribution implies that for any streamline

$$y/y_0 = Y/Y_0. \tag{21}$$

Now for the constant pressure on $-XOX$ (figure 3) we have

$$\Delta\rho g\bar{y}(x)+\int_0^{y_0(x)}\Delta\rho_y g\,dy = \int_0^{Y_0}\Delta\rho_y g\,dy+\Delta\rho g\bar{Y}. \tag{22}$$

Hence

$$\bar{y}(x) = [Y_0-y_0(x)]\int_0^1 f(\eta)\,d\eta+\bar{Y}. \tag{23}$$

Further, Bernoulli's equation for a streamline in the flow gives

$$\tfrac{1}{2}(\rho_0+\Delta\rho_y)v_y^2+\int_{y(x)}^{y_0(x)}\Delta\rho_y g\,dy+\Delta\rho_y g(\bar{y}(x)+y(x)) = \int_Y^{Y_0}\Delta\rho_y g\,dy+\Delta\rho_y g[\bar{Y}+Y] \tag{24}$$

and this becomes

$$\frac{(\rho_0+\Delta\rho_y)v_y^2}{\Delta\rho g(Y_0-y_0)} = 2\left[\int_\eta^1 f(\eta)\,d\eta+\eta f(\eta)-f(\eta)\int_0^1 f(\eta)\,d\eta\right] = 2g(\eta). \tag{25}$$

This then is the velocity distribution in terms of the density distribution for a similarity solution. It remains to determine the conditions under which this flow will go through the contraction.

Now let Q be the total discharge below the streamline defined by $y = \bar{y}(x)+y(x)$. Then the velocity on the streamline v_y may be written as

$$v_y = \frac{1}{b}\frac{dQ}{dy} = \frac{1}{by_0}\frac{dQ}{d\eta}$$

and this may be substituted into (25). The equation is then differentiated with respect to x and since the flow is self similar, $dQ/d\eta$ is independent of x and the equation becomes

$$\left(-2+\frac{y_0}{Y_0-y_0}\right)\frac{dy_0}{dx} = \frac{y_0}{b}\frac{db}{dx}. \tag{26}$$

When db/dx equals zero then either dy_0/dx equals zero or $y_0 = \frac{2}{3}Y_0$. As we are looking for a solution with dy_0/dx finite and y_0 continuously decreasing, it is only the second condition that is relevant. If now the Boussinesq assumption is made then the discharge is given by

$$Q = \left(\frac{2}{3}\frac{\Delta\rho}{\rho}gY_0\right)^{\frac{1}{2}}\left(\int_0^1 [g(\eta)]^{\frac{1}{2}}d\eta\right)\tfrac{2}{3}Y_0 b_c. \tag{27}$$

It is important to note that a continuously varying density stratification results in an infinite number of wave modes. Thus there are an infinite number of sections which are points of virtual control. The solution above is therefore the one which satisfies the condition that dy_0/dx is finite and negative at all of these points of virtual control.

4. The two-layer system and the departures from the self similar solution

It has already been noted that if the density difference between the lowest flowing layer and the underneath stationary fluid tends to infinity then this lowest streamline effectively becomes a solid horizontal bottom; it is now proposed to discuss this case further.

For the two-layer system there are two control points, one at the point of minimum width and one virtual control point further upstream and all the discharge calculations deal with values at these points. It is therefore possible to alter the geometry of the channel bed upstream of the virtual control point or downstream of the position of minimum width without affecting the flow through the contraction. That is, if before the virtual control point the reservoir is deepened and downstream of the point of minimum width the channel is steepened then the basic solution is unchanged and the flow becomes that of a two-layer system over a broad crested weir. With an increase in the number of layers the number of points of control increase and the broader the weir must become if the similarity solution over the weir is not to be disturbed.

Consider now the case of the flow of a two-layer system over a contracted weir with a relatively narrow crest. Here the point of virtual control will not fall on the same level as the crest. It would be most useful to be able to compute the discharges through this contracted weir for given upstream conditions and a given weir geometry. However, the amount of algebra involved in this type of calculation is formidable. The method in § 2 can, however, be used to obtain this information provided the discharges for the given upstream conditions are assumed to be those for the case where there is no weir in the contraction, and the weir geometry is treated as the unknown.

Define the vertical distance from a datum to the channel or reservoir bed as $h(x)$ and let $h'(x)$ be $h(x)/Y_1$. If at the section of maximum contraction dh/dx is zero, then from the differentiated Bernoulli's equation it can be shown that (8) again gives the conditions for the flow through the contraction to be smooth. However, the conditions that dy_1'/dx and dy_2'/dx are finite at the point of virtual control where equation (8) also applies are

$$\begin{vmatrix} -\dfrac{dh'}{dx}+F_1^2\dfrac{y_1'}{b}\dfrac{db}{dx} & 1 \\ -\dfrac{dh'}{dx}(1+\alpha_{12})+F_2^2\dfrac{y_2'}{b}\dfrac{db}{dx} & 1+\alpha_{12}-F_2^2 \end{vmatrix} = 0 \tag{28}$$

and
$$\begin{vmatrix} 1-F_1^2 & -\dfrac{dh'}{dx}+F_1^2\dfrac{y_1'}{b}\dfrac{db}{dx} \\ \alpha_{12} & -\dfrac{dh'}{dx}+F_2^2\dfrac{y_2'}{b}\dfrac{db}{dx} \end{vmatrix} = 0. \tag{29}$$

Now let it be assumed that the virtual control section occurs on a length of channel where the bed is level. Further let the level of this horizontal bed be taken as the datum from which all heights are measured. The equations at this virtual control point then reduce to those already solved, that is

$$c^2 = \frac{\alpha_{12}(1+Y_2')+Y_2'}{\rho_{21}\alpha_{12}(1+Y_2')}, \tag{11a}$$

$$y_2'/y_1' = Y_2', \tag{12a}$$

$$\phi = \frac{\rho_1 v_1^2}{2\Delta\rho_1 g Y_1} = \frac{[Y_2'(1+Y_2')]}{Y_2'^2(2\alpha_{12}+2)+Y_2'(4\alpha_{12}+3)+2\alpha_{12}}, \tag{13a}$$

$$y_1' = \frac{Y_2'^2(2\alpha_{12}+2)+Y_2'(4\alpha_{12}+2)+2\alpha_{12}}{Y_2'^2(2\alpha_{12}+2)+Y_2'(4\alpha_{12}+3)+2\alpha_{12}}. \tag{14a}$$

It now remains to satisfy the conditions at the position of the minimum width. Solving for y_1' and y_2' in terms of ϕ and c from Bernoulli's equations then the ratio of discharges at all points in the reservoir and at the position of maximum contraction may be written as

$$\frac{Q_2}{Q_1} = \frac{Y_2'[\alpha_{12}(1+Y_2')+Y_2']^{\frac{1}{2}}}{[\rho_{21}\alpha_{12}(1+Y_2')]^{\frac{1}{2}}} = c\left[\frac{\theta+\phi\alpha_{12}(1-\rho_{21}c^2)}{1-\phi[1+\alpha_{12}(1-\rho_{21}c^2)]}\right], \tag{30}$$

where $\theta = Y_2'-h_m'$ and h_m' is the maximum value of $h'(x)$. Further let the value of c at the point of virtual control be $\bar{c}$ and define $c_* = c/\bar{c}$, then the expression for ϕ that can be obtained from (30) is

$$\phi = \frac{Y_2'-c_*\theta}{c_*^3(-\alpha_{12}\rho_{21}\bar{c}^2)+c_*^2(-\alpha_{12}\rho_{21}\bar{c}^2 Y_2')+c_*\alpha_{12}+(1+\alpha_{12})Y_2'}. \tag{31}$$

Again using the expressions for y_1' and y_2' obtained from the Bernoulli equations, (8) becomes

$$\frac{[1-\phi[3+\alpha_{12}-\alpha_{12}\rho_{21}\bar{c}^2c_*^2]]}{[1-\phi[1+\alpha_{12}-\alpha_{12}\rho_{21}\bar{c}^2c_*^2]]}\frac{[\theta(1+\alpha_{12})+\alpha_{12}\phi[1+\alpha_{12}-\bar{c}^2c_*^2\rho_{21}(3+\alpha_{12})]]}{[\theta+\phi\alpha_{12}[1-\rho_{21}\bar{c}^2c_*^2]]} = \alpha_{12}. \tag{32}$$

Define $\rho_{21}\bar{c}^2 = \bar{c}_\rho^2$ and substitute for ϕ, then equation (32) becomes

$$\frac{\theta[-c_*^3\bar{c}_\rho^2\alpha_{12}+c_*(3+\alpha_{12})]-\bar{c}_\rho^2\alpha_{12}c_*^3+\alpha_{12}c_*-2Y_2'}{\theta[-c_*^3\bar{c}_\rho^2\alpha_{12}+c_*(1+\alpha_{12})]-\bar{c}_\rho^2\alpha_{12}c_*^3+c_*\alpha_{12}}\times$$

$$\frac{\theta[2\alpha_{12}\bar{c}_\rho^2c_*^3-(1+\alpha_{12})\alpha_{12}\bar{c}_\rho^2c_*^2Y_2'+Y_2(1+\alpha_{12})^2] -\bar{c}_\rho^2c_*^2Y_2'\alpha_{12}(3+\alpha_{12})+Y_2'\alpha_{12}(1+\alpha_{12})}{\theta[-c_*^2\bar{c}_\rho^2Y_2'+Y_2'(1+\alpha_{12})]+Y_2'\alpha_{12}-\bar{c}_\rho^2c_*^2Y_2'} = \alpha_{12}. \tag{33}$$

If c_* at the position of minimum width is assumed known, then the above equation is a simple quadratic and may be solved for θ. For a given c_* the two solutions for θ, represent a very small drop, and quite a large rise in bed level at the contraction. The latter case is physically more interesting. Once this value has been determined, values of y_1', y_2' and ϕ at the position of minimum width can be calculated from the Bernoulli equations and (31). Finally, using the equation of continuity the width at the contraction can be computed. The calculations can be carried out until the width at the contraction equals that at the point of virtual control.

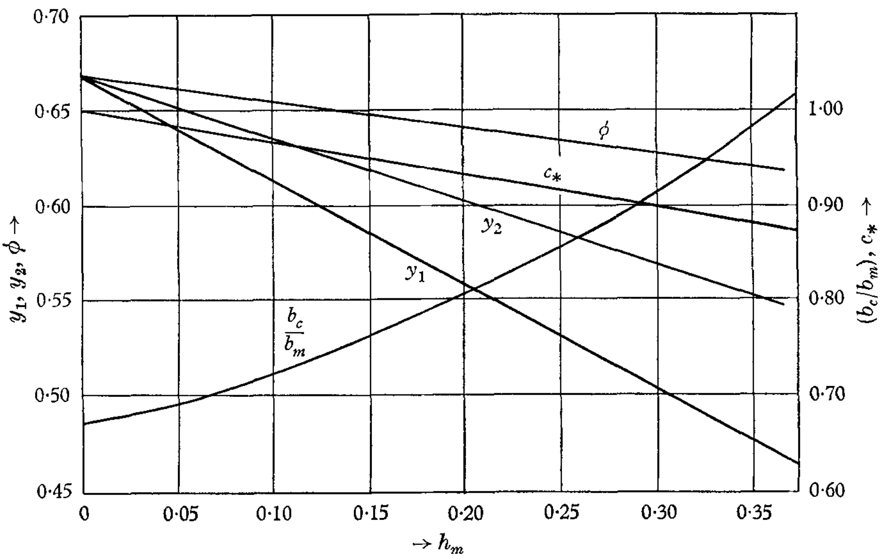

FIGURE 4. The characteristics of the flow of a two-layer system through a narrow crested contracted weir. $\rho_{21} = 1$, $\alpha_{12} = 1$, $Y_2 = 1$.

For the case where ρ_{21} equals one (this is equivalent to the Boussinesq assumption) and for upstream conditions of $Y_2' = 1$ and $\alpha_{12} = 1$ the ratios of the minimum width to that at the virtual control points, y_1', y_2' and ϕ are plotted against the height of the rise in contraction, in figure 4.

It seems likely that this method can be extended to systems containing more than two flowing layers but the amount of algebra involved increases considerably.

5. Experimental verification

A number of preliminary experiments were carried out in a $365 \times 7{\cdot}5 \times 10$ cm closed tunnel for the case where the lowermost streamline was a solid boundary. The tunnel could be tilted about an axis parallel to the 7·5 cm dimension and in order to reduce the length of interface over which mixing was possible the tunnel was always filled in the tilted position. A horizontal contraction was placed in the tunnel and a plan of this contraction is shown in figure 5. In every case salt in water was used to obtain the required density differences. For the layered case the tunnel was filled at a very slow rate and the central layer was dyed. For the case of the linear density gradient the method of filling used was that described by Oster (1965). After filling, patches of dye were injected through nozzles in the roof of the tunnel. The tunnel was then slowly lowered into a horizontal position. In both cases negligible mixing was caused by this slow lowering and in the case of the linear density gradient the patches of dye were stretched giving fine horizontal lines of dye. Air holes were then opened on the top of the tunnel and sufficient fluid was withdrawn from the 7·5 cm × 1 mm slit at the outlet end of the tunnel so that the fluid within the tunnel had a free surface. Sufficient time was then allowed for all the velocities within the tunnel to decay and then the experiment was started.

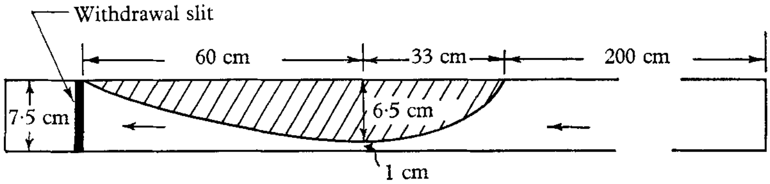

FIGURE 5. Plan of the significant dimensions of the main contraction (a second contraction was used and this had a minimum width of 2 cm).

Withdrawal commenced and the discharge was measured with a rotameter and was maintained at a constant value throughout the experiment. Photographs were taken at frequent intervals and from these the depths of the layers in the upstream portion of the channel and at the point of maximum contraction were obtained.

The case of the flow with an originally linear density gradient was the easier to analyse. In this case as the upstream boundary conditions gradually changed the number of dye lines with a shape indicating a flow through the contraction increased (figure 6, plate 1). As each dye streak changed from its originally horizontal position to one indicating a flow through the contraction and out of the sink it induced small velocities in the region above it. These were not associated with the flow through the contraction but were necessary to form the wedges of constant density fluid above the flowing region. These velocities were small enough not to affect the measurements of depths but made measurements taken from the vertical dye streaks (figure 6) of doubtful value.

Now for each streamline which indicated a smooth flow through the contraction, theory shows that the ratio of its depth at the position of minimum width to its depth in the upstream reservoir should be $\frac{2}{3}$. In each photograph of the experiment a different number of the originally horizontal dye streaks indicated streamlines which flowed smoothly through the contraction. For each of these dye streaks the ratio of the depth at the contraction to that far upstream was computed. From six of the photographs taken approximately 5 sec apart 26 separate measurements of the ratio were made and gave a mean and a standard deviation of the mean of $0{\cdot}65 \pm 0{\cdot}01$.

The interpretation of the experiments with a discretely layered system was considerably more difficult. If the valve was slowly opened then the lowest layer commenced to flow. As it was opened still further there was flow in the lowest two layers and finally when the valve was opened still further the discharge came from all three layers. There was obviously a range of discharges at which there are flows in the lower two layers only but the theory indicates that there is only one discharge at which there is flow in only the lower two layers and the depths of the layers decrease continuously from their reservoir depth to the channel depth. For this discharge the flow is controlled at and upstream of the channel contraction. When the discharge is less than this critical discharge there appears to be a weak internal hydraulic jump downstream of the contraction and the control at the contraction is effectively drowned. When the discharge was greater than the critical one then withdrawal has commenced from the uppermost layer.

The time taken for the system to react to valve changes compared to the time for change in the upstream reservoir made it inappropriate to operate an experiment as described above.

The discharge was therefore held constant throughout each experiment and the slow changes in the upstream condition caused slow changes in the flow through the contraction. When flow commenced in the middle layer the weak hydraulic jump was observed but as the upstream conditions slowly changed, the jump moved downstream and its downstream depth decreased. Depth measurements were taken when the downstream depth of the hydraulic jump (y_1+y_2) was less than 0·7 times the upstream reservoir depths (Y_1+Y_2). At this stage the velocities in the uppermost layers were small enough not to affect the depth of flow at the contraction and a comparison with open channel flow measurements suggested that the jump would no longer control the flow. The experiments were carried out with the density ratio α_{12} ranging from 1 to 0·25† and an upstream *depth* ratio ranging from 1 to $\frac{1}{2}$. These yielded ten measurements giving a mean and the standard deviation of the mean of $0{\cdot}65 \pm 0{\cdot}02$.

In all of the experiments there were small time dependent effects and small circulations were set up in the overlying fluid. These circulations were caused by (i) the flow into the wedge of constant density fluid (figures 3 and 6) and (ii) the effect of viscosity. Because of these small circulations no serious attempt was made to measure velocity distributions within the system. However, observations with dye streaks did indicate the correct trends in the velocity distributions. All of the above effects could be minimized by using larger equipment and further experiments are planned.

The effects do not greatly affect the measurements of the depth and the agreement between theory and experiment must be regarded as satisfactory.

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† If the well verified case of a uniform fluid is included, this density ratio range is from ∞ to 0·25.

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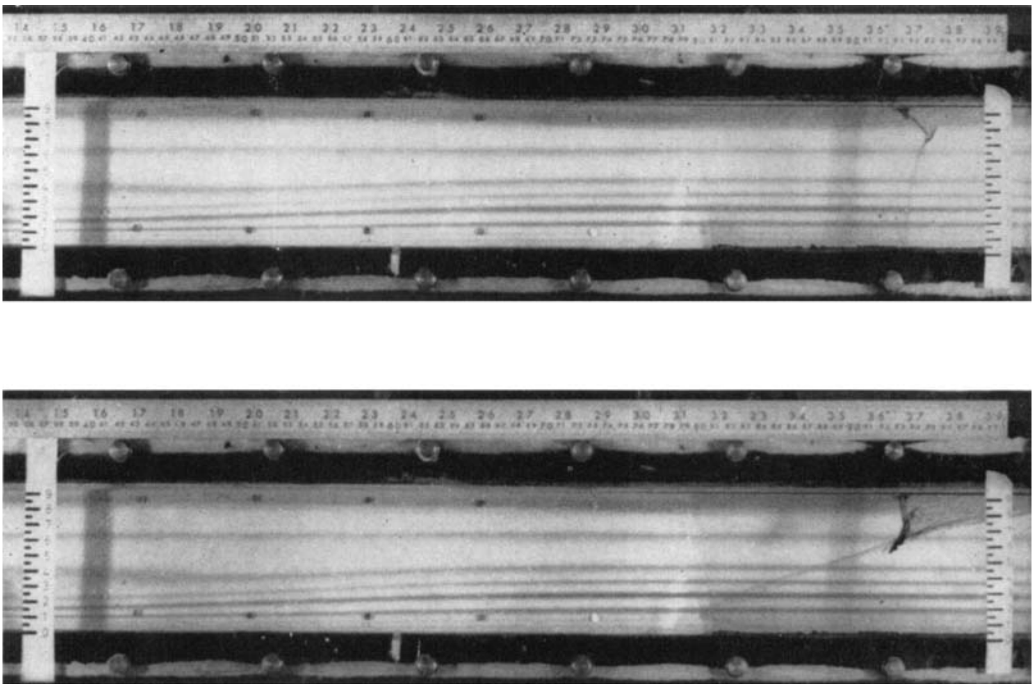

FIGURE 6. Photographs taken approximately one second apart of the flow of a fluid with a stable density gradient through a contraction. (The position of minimum width is at 19·75 inches on the scale.) The withdrawal slit is on the bottom of the tunnel at a scale distance of 11 in.

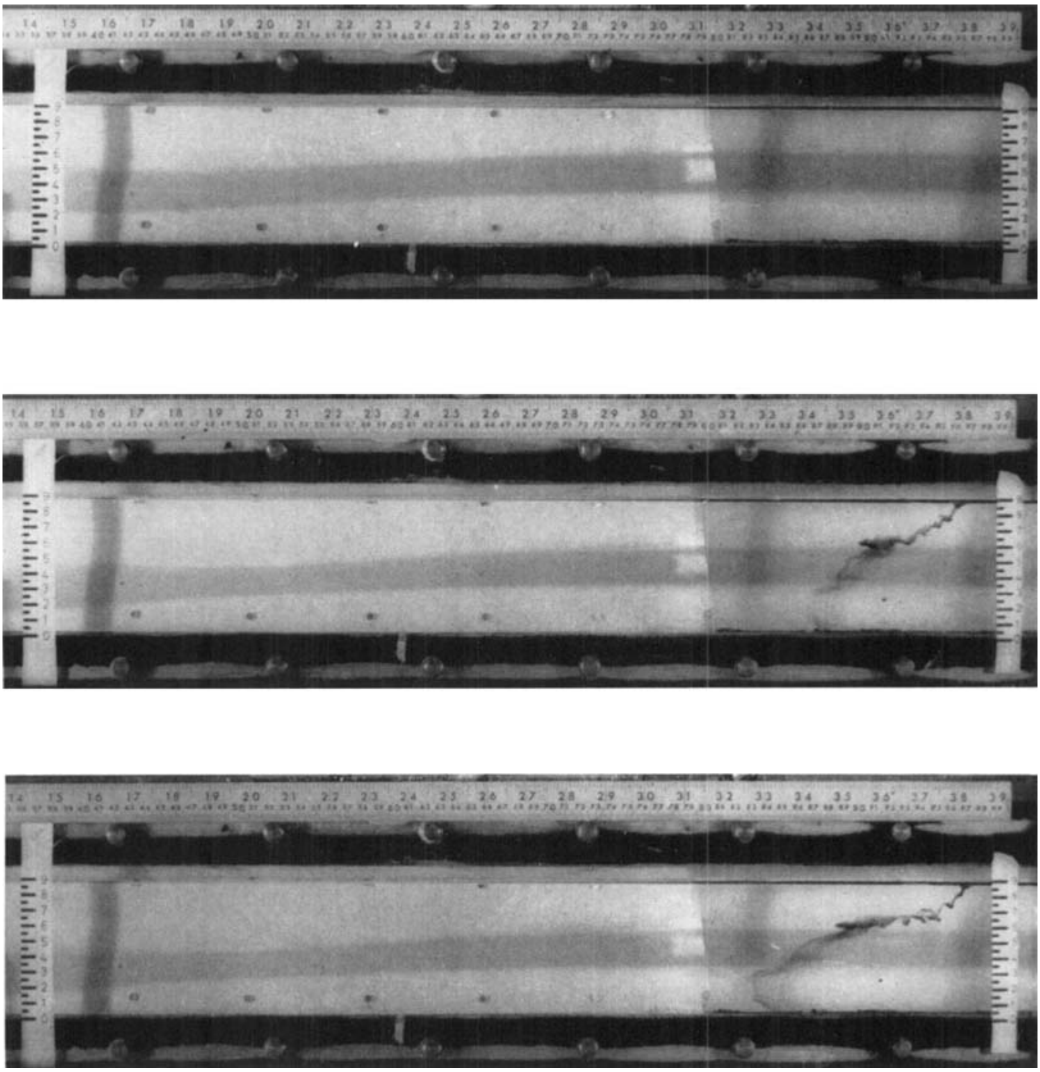

FIGURE 7. Photographs taken at approximately two seconds apart of the flow of a two fluid system through a contraction. (The position of minimum width is at 19·75 in. on the scale.) The withdrawal slit is on the bottom of the tunnel at a scale distance of 11 in.